

Renewal theorems and mixing for non Markov flows with infinite measure

Ian Melbourne ^{*} Dalia Terhesiu [†]

29 January 2017

Abstract

We obtain results on mixing for a large class of (not necessarily Markov) infinite measure semiflows and flows. Erickson proved, amongst other things, a strong renewal theorem in the corresponding i.i.d. setting. Using operator renewal theory, we extend Erickson's methods to the deterministic (i.e. non-i.i.d.) continuous time setting and obtain results on mixing as a consequence.

1 Introduction

Recently, there has been increasing interest in the investigation of mixing properties for infinite measure-preserving dynamical systems [2, 13, 22, 23, 24, 27, 28, 29, 30, 31, 32, 35, 36, 38]. Most of these results are for discrete time noninvertible systems.

For results on semiflows preserving an infinite measure, we refer to [32] (the Markov case) and [13] (which does not assume a Markov structure). The setting is that $F : Y \rightarrow Y$ is a mixing uniformly expanding map defined on a probability space (Y, μ) and $\tau : Y \rightarrow \mathbb{R}^+$ is a nonintegrable roof function with regularly varying tails:

$$\mu(y \in Y : \tau(y) > t) = \ell(t)t^{-\beta} \quad \text{for various ranges of } \beta \in [0, 1]. \quad (1.1)$$

Here, $\ell : [0, \infty) \rightarrow [0, \infty)$ is a measurable slowly varying function (so $\lim_{t \rightarrow \infty} \ell(\lambda t)/\ell(t) = 1$ for all $\lambda > 0$). Consider the suspension (Y^τ, μ^τ) and suspension semiflow $F_t : Y^\tau \rightarrow Y^\tau$ (the standard definitions are recalled in Section 3). The aim is to prove a mixing result of the form

$$\lim_{t \rightarrow \infty} a_t \int_{Y^\tau} v w \circ F_t d\mu^\tau = \int_{Y^\tau} v d\mu^\tau \int_{Y^\tau} w d\mu^\tau,$$

^{*}Mathematics Institute, University of Warwick, Coventry, CV4 7AL, UK

[†]Department of Mathematics, University of Exeter, Exeter EX4 4QF, UK

for a suitable normalisation $a_t \rightarrow \infty$ and suitable classes of observables $v, w : Y^\tau \rightarrow \mathbb{R}$.

Under certain hypotheses, [13, 32] obtained results on mixing and rates of mixing for such semiflows. The hypotheses were of two types: (i) assumptions on “renewal operators” associated to the transfer operator of F and the roof function τ , and (ii) Dolgopyat-type assumptions of the type used to obtain mixing rates for finite measure (semi)flows [16].

As pointed out to us by Dima Dolgopyat, Péter Nándori and Doma Szász, mixing for indicator functions can be regarded as a local limit theorem and hence hypotheses of type (ii) should not be necessary.

In this paper, we show that operator renewal-theoretic assumptions (i) are indeed sufficient for obtaining the mixing results in [13, 32]. The abstract framework in [13] turns out again to be flexible enough to cover nonMarkov situations. Moreover, our main results extend to flows and we are able to treat large classes of observables v, w . (The methods in this paper are not appropriate for obtaining rates of mixing; the best results currently are those in [13].)

The analogous probabilistic results go back to Erickson [19] who obtained *strong renewal theorems* in an i.i.d. continuous time framework under the assumption $\beta \in (\frac{1}{2}, 1]$. (In the discrete time setting, see [21] for the i.i.d. case and [31] for the deterministic case.) Our results on mixing when $\beta \in (\frac{1}{2}, 1]$ for semiflows (Corollary 3.1 and the extensions in Section 7) and for flows (Theorem 8.5), are proved by adapting Erickson’s methods to the deterministic setting.

For $\beta \leq \frac{1}{2}$, additional hypotheses are needed on the tail of τ to obtain a strong renewal theorem (and hence mixing) even for discrete time; see [15, 18, 21] for i.i.d. results and [22] for deterministic results (see also [36] for higher order theory in both the i.i.d. and deterministic settings). For the continuous time case, in work in progress, Dolgopyat & Nándori [17] obtain strong renewal theorems for a class of Markov semiflows with $\beta \leq \frac{1}{2}$ (again under extra hypotheses on the tail $\mu(\tau > t)$). In the absence of additional tail hypotheses, [19] showed how to obtain a partial result in the probabilistic setting with limit replaced by \liminf . In Corollary 3.5, we obtain such a \liminf result for semiflows with $\beta \in (0, \frac{1}{2}]$.

The remainder of this paper is organised as follows. In Section 2, we describe the operator renewal-theoretic hypotheses required in this paper and we state a strong renewal theorem for $\beta \in (\frac{1}{2}, 1]$ as well as related results for $\beta \leq \frac{1}{2}$. In Section 3, we show how these results lead to mixing properties for semiflows. Sections 4 and 5 are devoted to the proof of the strong renewal theorem, and Section 6 contains the proofs of the remaining results in Section 2.

Corollary 3.1 (mixing for semiflows) is stated for observables that are certain indicator functions. This restriction is relaxed considerably in Section 7. The corresponding result for flows is stated and proved in Section 8. Illustrative examples are given in Section 9.

2 Strong renewal theorem for continuous time deterministic systems

Let (Y, μ) be a probability space and let $F : Y \rightarrow Y$ be an ergodic and mixing measure-preserving transformation. Let $\tau : Y \rightarrow \mathbb{R}^+$ be a measurable nonintegrable function bounded away from zero. For convenience, we suppose that $\text{ess inf } \tau > 1$. Throughout we assume the regularly varying tail condition (1.1).

Let $\tau_n = \sum_{j=0}^{n-1} \tau \circ F^j$. Given measurable sets $A, B \subset Y$, define the renewal measure

$$U_{A,B}(I) = \sum_{n=0}^{\infty} \mu(y \in A \cap F^{-n}B : \tau_n(y) \in I), \quad (2.1)$$

for any interval $I \subset \mathbb{R}$. We write $U_{A,B}(x) = U_{A,B}([0, x])$ for $x > 0$. Our aim is to generalise [19, Theorems 1 and 2] to this set up. That is, we want to obtain the asymptotics of $U_{A,B}(t+h) - U_{A,B}(t)$ for any $h > 0$.

With the same notation as in [13], let $\overline{\mathbb{H}} = \{\text{Re } s \geq 0\}$. Given $\delta > 0$ and $L > 0$, let $\mathbb{H}_{\delta,L} = (\overline{\mathbb{H}} \cap B_\delta(0)) \cup \{ib : |b| \leq L\}$. Define the family of operators for $s \in \overline{\mathbb{H}}$,

$$\hat{R}(s) : L^1(Y) \rightarrow L^1(Y), \quad \hat{R}(s)v = R(e^{-s\tau}v).$$

Here $R : L^1(Y) \rightarrow L^1(Y)$ is the transfer operator for F (so $\int_Y Rv w d\mu = \int_Y v w \circ F d\mu$ for all $v \in L^1(Y)$, $w \in L^\infty(Y)$).

Let $\mathcal{B} = \mathcal{B}(Y)$ be a Banach space containing constant functions, with norm $\|\cdot\|_{\mathcal{B}}$. We assume that for any $L > 0$, there exist constants $\delta > 0$, $\alpha_0 \in (0, 1)$ and $C > 0$, such that

- (H) (i) $\|\hat{R}(s)^n v\|_{\mathcal{B}} \leq C(|v|_1 + \alpha_0^n \|v\|_{\mathcal{B}})$ for all $s \in \mathbb{H}_{\delta,L}$, $v \in \mathcal{B}$, $n \geq 1$.
- (ii) The operator $R : \mathcal{B} \rightarrow \mathcal{B}$ has a simple eigenvalue at 1 and the rest of the spectrum is contained in a disk of radius less than 1.
- (iii) \mathcal{B} is embedded in L^p for some $p > \beta/(2\beta - 1)$, and is compactly embedded in L^1 .
- (iv) The spectrum of $\hat{R}(ib) : \mathcal{B} \rightarrow \mathcal{B}$ does not contain 1 for all $b \in \mathbb{R} \setminus \{0\}$.

Hypothesis (H)(i)–(iii) is a slight strengthening of [13, hypothesis (H1)] (where $L = \delta$ is fixed and arbitrarily small, and the first part of (iii) is in the statement of the main results in [13]). (H)(iv) is a significant weakening of [13, hypothesis (H4)] and the diophantine ratio assumption used in [32] (Dolgopyat-type condition). The remaining hypotheses in [13], namely (H2) and (H3) (re-inducing) are not required.

Remark 2.1 As explained in Section 9, hypothesis (H) is satisfied for mixing Gibbs-Markov maps (with \mathcal{B} a symbolic Hölder space) and mixing AFU maps (with \mathcal{B} consisting of bounded variation functions) provided, for instance, there exist two periodic orbits for F_t with periods q_1, q_2 such that q_1/q_2 is irrational. In both cases, we can take $p = \infty$.

Remark 2.2 To verify the Lasota-Yorke inequality in (H)(i), it suffices to proceed as follows. (a) Verify it for $s = 0$. (b) Verify that the family of operators $s \mapsto \hat{R}(s)$ on \mathcal{B} is continuous at $s = 0$. (c) Show that for every $L > 0$, there exists C_0, n_0 such that $\|\hat{R}(ib)\|_{\mathcal{B}} \leq C_0$ and $\|\hat{R}(ib)^{n_0}v\|_{\mathcal{B}} \leq C_0|v|_1 + \frac{1}{2}\|v\|_{\mathcal{B}}$ for all $v \in \mathcal{B}$, $|b| \leq L$.

Define $d_\beta = \frac{1}{\pi} \sin \beta\pi$, $m(t) = \ell(t)t^{1-\beta}$ for $\beta < 1$, and $d_\beta = 1$, $m(t) = \int_1^t \ell(s)s^{-1} ds$ for $\beta = 1$. Throughout we suppose that $A, B \subset Y$ are measurable and that $1_A \in \mathcal{B}$.

Our main result generalizes [19, Theorem 1] to the present non i.i.d. set up:

Theorem 2.3 (Strong renewal theorem) *Assume $\mu(\tau > t) = \ell(t)t^{-\beta}$ where $\beta \in (\frac{1}{2}, 1]$. Suppose that (H) holds. Then for any $h > 0$,*

$$\lim_{t \rightarrow \infty} m(t)(U_{A,B}(t+h) - U_{A,B}(t)) = d_\beta \mu(A)\mu(B)h.$$

As discussed in the introduction, additional hypotheses are needed to obtain a strong renewal theorem when $\beta \leq \frac{1}{2}$. However, generalizing [19, Theorem 2] to the present non i.i.d. set up, we still obtain a lim inf result.

Theorem 2.4 *Assume $\mu(\tau > t) = \ell(t)t^{-\beta}$ where $\beta \in (0, 1)$. Suppose that (H) holds. Then for any $h > 0$,*

$$\liminf_{t \rightarrow \infty} m(t)(U_{A,B}(t+h) - U_{A,B}(t)) = d_\beta \mu(A)\mu(B)h.$$

Remark 2.5 In the i.i.d. setting, results of this type are first due to [21] for discrete time and $\beta < 1$. The results of [19] extended [21] to continuous time and incorporated the case $\beta = 1$.

For the proof of Theorem 2.4, we will need the following result which gives the asymptotics of $U(t)$ for the whole range $\beta \in [0, 1]$. This implies a property for the semiflow F_t known as *weak rational ergodicity* [1, 4] (see Corollary 3.3 below) and thus is of interest in its own right.

Theorem 2.6 *Assume $\mu(\tau > t) = \ell(t)t^{-\beta}$ where $\beta \in [0, 1]$. Suppose that (H)(i) holds for $s \in \mathbb{R}^+$ and that (H)(ii,iii) hold. Then*

$$\lim_{t \rightarrow \infty} t^{-1}m(t)U_{A,B}(t) = D_\beta \mu(A)\mu(B),$$

where $D_\beta = [\Gamma(1-\beta)\Gamma(1+\beta)]^{-1}$, if $\beta \in (0, 1)$ and $D_0 = D_1 = 1$.

3 Mixing for infinite measure semiflows

In this section, we obtain various mixing results for semiflows as consequences of the results in Section 2.

Let $F : Y \rightarrow Y$ and $\tau : Y \rightarrow \mathbb{R}^+$ be as in Section 2. Define the suspension $Y^\tau = \{(y, u) \in Y \times \mathbb{R} : 0 \leq u \leq \tau(y)\} / \sim$ where $(y, \tau(y)) \sim (Fy, 0)$. The suspension semiflow $F_t : Y^\tau \rightarrow Y^\tau$ is given by $F_t(y, u) = (y, u + t)$, computed modulo identifications. The measure $\mu^\tau = \mu \times \text{Lebesgue}$ is ergodic, F_t -invariant and σ -finite. Since τ is nonintegrable, μ^τ is an infinite measure.

Throughout this section, we suppose that $A_1 = A \times [a_1, a_2]$, $B_1 = B \times [b_1, b_2]$ are measurable subsets of $\{(y, u) \in Y \times \mathbb{R} : 0 \leq u \leq \tau(y)\}$ (so $0 \leq a_1 < a_2 \leq \text{ess inf}_A \tau$, $0 \leq b_1 < b_2 \leq \text{ess inf}_B \tau$), and that $1_A \in \mathcal{B}$. Also, we continue to suppose that $\mu(\tau > t) = \ell(t)t^{-\beta}$ for various ranges of $\beta \in [0, 1]$.

Corollary 3.1 *Assume the setting of Theorem 2.3, so $\beta \in (\frac{1}{2}, 1]$ and (H) holds. Then $\lim_{t \rightarrow \infty} m(t)\mu^\tau(A_1 \cap F_t^{-1}B_1) = d_\beta\mu^\tau(A_1)\mu^\tau(B_1)$.*

Proof Recall that $\text{ess inf } \tau > 1$. Let $h \in (0, 1)$ and note using (2.1) that

$$\begin{aligned} U_{A,B}(t+h) - U_{A,B}(t) &= \mu(y \in A : F^n y \in B \text{ and } \tau_n(y) \in [t, t+h] \text{ for some } n \geq 0) \\ &= \mu(y \in A : F_{t+h}(y, 0) \in B \times [0, h]). \end{aligned}$$

After dividing rectangles into smaller subrectangles, we can suppose without loss that $b_2 - b_1 < 1$. Set $h = b_2 - b_1$. Then

$$\begin{aligned} \mu^\tau(A_1 \cap F_t^{-1}B_1) &= \mu^\tau\{(y, u) \in A \times [a_1, a_2] : F_t(y, u) \in B \times [b_1, b_2]\} \\ &= \mu^\tau\{(y, u) \in A \times [a_1, a_2] : F_{t+u-b_1}(y, 0) \in B \times [0, h]\} \\ &= \int_{a_1}^{a_2} \mu\{y \in A : F_{t+u-b_1}(y, 0) \in B \times [0, h]\} du \\ &= \int_{a_1}^{a_2} (U_{A,B}(t+u-b_1) - U_{A,B}(t+u-b_1-h)) du. \end{aligned} \quad (3.1)$$

Hence

$$\begin{aligned} m(t)\mu^\tau(A_1 \cap F_t^{-1}B_1) &= \int_{a_1}^{a_2} m(t)(U_{A,B}(t+u-b_1) - U_{A,B}(t+u-b_1-h)) du \\ &= \int_{a_1}^{a_2} \frac{m(t)}{m(t+u-b_1-h)} \chi(t+u-b_1-h) du, \end{aligned}$$

where $\chi(t) = m(t)(U_{A,B}(t+h) - U_{A,B}(t))$ is bounded by Theorem 2.3. Also $m(t)/m(t+u-b_1-h)$ is bounded by Potter's bounds (see for instance [11]). Since $m(t)$ is regularly varying, we have $\lim_{t \rightarrow \infty} m(t)/m(t+u-b_1-h) = 1$ for each $u \in [0, 1]$. By Theorem 2.3, $\lim_{t \rightarrow \infty} \chi(t+u-b_1-h) = d_\beta\mu(A)\mu(B)h = d_\beta\mu(A)\mu^\tau(B_1)$ for each $u \in [0, 1]$. Hence the result follows from the bounded convergence theorem. \blacksquare

Remark 3.2 The result also holds for all sets of the form $F_r^{-1}A_1$ and $F_s^{-1}B_1$ for fixed $r, s > 0$. Indeed, by Corollary 3.1, using that $m(t) \sim m(t+s-r)$,

$$\begin{aligned} m(t)\mu^\tau(F_r^{-1}A_1 \cap F_{t+s}^{-1}B_1) &= m(t)\mu^\tau(A_1 \cap F_{t+s-r}^{-1}B_1) \\ &\rightarrow \mu^\tau(A_1)\mu^\tau(A_2) = \mu^\tau(F_r^{-1}A_1)\mu^\tau(F_s^{-1}A_2). \end{aligned}$$

Corollary 3.3 (Weak rational ergodicity) *Assume the setting of Theorem 2.6, so $\beta \in [0, 1]$, $(H)(i)$ holds for $s \in \mathbb{R}^+$, and $(H)(ii, iii)$ hold. Then*

$$\lim_{t \rightarrow \infty} t^{-1} m(t) \int_0^t \mu^\tau(A_1 \cap F_x^{-1} B_1) dx = D_\beta \mu^\tau(A_1) \mu^\tau(B_1).$$

Proof Continuing from (3.1) (with $h = b_2 - b_1$),

$$\begin{aligned} \int_0^t \mu^\tau(A_1 \cap F_x^{-1} B_1) dx &= \int_{a_1}^{a_2} \int_0^t (U_{A,B}(x+u-b_1) - U_{A,B}(x+u-b_1-h)) dx du \\ &= \int_{a_1}^{a_2} \int_0^t U_{A,B}(x+u-b_1) dx du - \int_{a_1}^{a_2} \int_{-h}^{t-h} U_{A,B}(x+u-b_1) dx du \\ &= \int_{a_1}^{a_2} \int_{t-h}^t U_{A,B}(x+u-b_1) dx du - \int_{a_1}^{a_2} \int_{-h}^0 U_{A,B}(x+u-b_1) dx du = I_1 + I_2. \end{aligned}$$

Now

$$t^{-1} m(t) I_1 = t^{-1} m(t) U_{A,B}(t) \int_{a_1}^{a_2} \int_{-h}^0 \frac{U_{A,B}(x+t+u-b_1)}{U_{A,B}(t)} dx du.$$

By Theorem 2.6, $U_{A,B}(t)$ is regularly varying so the integrand $U_{A,B}(x+t+u-b_1)/U_{A,B}(t)$ is bounded for x, u bounded and converges pointwise to 1 as $t \rightarrow \infty$. Hence

$$\lim_{t \rightarrow \infty} \int_{a_1}^{a_2} \int_{-h}^0 \frac{U_{A,B}(x+t+u-b_1)}{U_{A,B}(t)} dx du = (a_2 - a_1)h = (a_2 - a_1)(b_2 - b_1).$$

By Theorem 2.6, $t^{-1} m(t) U_{A,B}(t) = D_\beta \mu(A) \mu(B) (1 + o(1))$. Hence, $\lim_{t \rightarrow \infty} t^{-1} m(t) I_1 = D_\beta \mu(A) \mu(B) (a_2 - a_1)(b_2 - b_1) = \mu^\tau(A_1) \mu^\tau(B_1)$. A simpler argument shows that $t^{-1} m(t) I_2 = o(1)$. \blacksquare

Proposition 3.4 *Let $f : [0, \infty) \rightarrow \mathbb{R}$ be bounded and integrable on compact sets, and let $K \in \mathbb{R}$. Suppose that $\beta \in (0, 1)$, that $\ell(t)$ is slowly varying, and that*

- (a) $\liminf_{t \rightarrow \infty} \ell(t) t^{1-\beta} f(t) \geq K$,
- (b) $\lim_{t \rightarrow \infty} \ell(t) t^{-\beta} \int_0^t f(x) dx = \beta^{-1} K$.

Then there exists a set $E \subset [0, \infty)$ of density zero such that $\lim_{t \rightarrow \infty, t \notin E} \ell(t) t^{1-\beta} f(t) = K$.

In particular, $\liminf_{t \rightarrow \infty} \ell(t) t^{1-\beta} f(t) = K$.

Proof This is the continuous time analogue of [31, Proposition 8.2] (which is itself a version of [33, p. 65, Lemma 6.2]). We list the main steps which are proved exactly as in [31].

Step 1. Without loss of generality, $K = 0$ and $\ell(t)t^{1-\beta}$ is increasing.

Step 2. Define the nested sequence of sets $E_q = \{t > 0 : \ell(t)t^{1-\beta}f(t) > 1/q\}$, $q = 1, 2, \dots$. Then E_q has density zero for each q , i.e. $\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t 1_{E_q}(x) dx = 0$.

Step 3. By Step 2, we can choose $0 = i_0 < i_1 < i_2 < \dots$ such that $\frac{1}{t} \int_0^t 1_{E_q}(x) dx < 1/q$ for $t \geq i_{q-1}$, $q \geq 2$. Define $E = \bigcup_{q=1}^{\infty} E_q \cap (i_{q-1}, i_q)$. Then E has density zero and $\lim_{t \rightarrow \infty, t \notin E} \ell(t)t^{1-\beta}f(t) = 0$. ■

Corollary 3.5 *Assume the setting of Theorem 2.4, so $\beta \in (0, 1)$ and (H) holds. Then*

- (i) $\liminf_{t \rightarrow \infty} m(t)\mu^\tau(A_1 \cap F_t^{-1}B_1) = d_\beta\mu^\tau(A_1)\mu^\tau(B_1)$, and
- (ii) *There exists a set $E \subset [0, \infty)$ of density zero such that*
 $\lim_{t \rightarrow \infty, t \notin E} m(t)\mu^\tau(A_1 \cap F_t^{-1}B_1) = d_\beta\mu^\tau(A_1)\mu^\tau(B_1)$.

Proof We start from the conclusion of Theorem 2.4. Arguing as in the proof of Corollary 3.1, but with \lim replaced by \liminf and using Fatou's lemma instead of the bounded convergence theorem, we obtain

$$\liminf_{t \rightarrow \infty} \ell(t)t^{1-\beta}\mu^\tau(A_1 \cap F_t^{-1}B_1) \geq d_\beta\mu^\tau(A_1)\mu^\tau(B_1).$$

This is condition (a) in Proposition 3.4, and Corollary 3.3 is condition (b). Hence the result follows from Proposition 3.4. ■

4 Main results used in the proof of Theorem 2.3

The first result needed in the proof of the strong renewal theorem, Theorem 2.3, is an inversion formula for the symmetric measure

$$V_{A,B}(I) = \frac{1}{2}(U_{A,B}(I) + U_{A,B}(-I)).$$

Here, $U(-I) = U(\{x : -x \in I\})$ (with $U(-I) = 0$ if $I \subset [0, \infty]$). We find it convenient to adapt the formulation in [19, Section 4], but such an inversion formula goes back to [20] (see also [12, Chapter 10]).¹

By (H), $\hat{T}(s) = (I - \hat{R}(s))^{-1}$ is a bounded operator on \mathcal{B} for all $s \in \overline{\mathbb{H}} \setminus \{0\}$. Let $A, B \subset Y$ be measurable with $1_A \in \mathcal{B}$.

Proposition 4.1 (Analogue of [19, Inversion formula, Section 4].) *Let $g : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous compactly supported function with Fourier transform $\gamma(x) = \int_{-\infty}^{\infty} e^{ixb}g(b)db$ satisfying $\gamma(x) = O(x^{-2})$ as $x \rightarrow \infty$. Then for all $\lambda, t \in \mathbb{R}$,*

$$\int_{-\infty}^{\infty} e^{-i\lambda(x-t)}\gamma(x-t)dV_{A,B}(x) = \int_{-\infty}^{\infty} e^{-itb}g(b+\lambda)\operatorname{Re} \int_B \hat{T}(ib)1_A d\mu db.$$

¹The result does not require any regular variation assumptions on $\mu(\tau > t)$, but we use the extra structure for simplicity.

The second result required in the proof of Theorem 2.3 comes directly from [19] and does not require any modification in our set up. To state this result, for each $a > 0$ we let $\gamma_a(0) = 1$ and for $x \neq 0$, define

$$\gamma_a(x) = \frac{2(1 - \cos ax)}{a^2 x^2}.$$

Proposition 4.2 ([19, Lemma 8]) *Let $\{\mu_t, t > 0\}$ be a family of measures such that $\mu_t(I) < \infty$ for every compact set I and all t . Suppose that for some constant C ,*

$$\lim_{t \rightarrow \infty} \int_{-\infty}^{\infty} e^{-i\lambda x} \gamma_a(x) d\mu_t(x) = C \int_{-\infty}^{\infty} e^{-i\lambda x} \gamma_a(x) dx,$$

for all $a > 0$, $\lambda \in \mathbb{R}$. Then $\mu_t(I) \rightarrow C|I|$ for every bounded interval I , where $|I|$ denotes the length of I . \blacksquare

Next, note that γ_a is the Fourier transform of

$$g_a(b) = \begin{cases} a^{-1}(1 - |b|/a), & |b| \leq a \\ 0, & |b| > a \end{cases}.$$

The final result required in the proof of Theorem 2.3 is as follows.

Proposition 4.3 *For all $a > 0$ and $\lambda \in \mathbb{R}$,*

$$\lim_{t \rightarrow \infty} m(t) \int_{-\infty}^{\infty} e^{-itb} g_a(b + \lambda) \operatorname{Re} \int_B \hat{T}(ib) 1_A d\mu db = \pi d_\beta g_a(\lambda) \mu(A) \mu(B).$$

Proof of Theorem 2.3 With the convention $I + t = \{x : x - t \in I\}$, let

$$\mu_t(I) = 2m(t)V_{A,B}(I + t) = m(t)(U_{A,B}(I + t) + U_{A,B}(-I - t))$$

and note that for $I = [0, h]$ with $h > 0$,

$$m(t)(U_{A,B}(t + h) - U_{A,B}(t)) = \mu_t(I).$$

Now,

$$\begin{aligned} m(t) \int_{-\infty}^{\infty} e^{-i\lambda(x-t)} \gamma_a(x-t) dV_{A,B}(x) &= m(t) \int_{-\infty}^{\infty} e^{-i\lambda x} \gamma_a(x) dV_{A,B}(x+t) \\ &= \frac{1}{2} \int_{-\infty}^{\infty} e^{-i\lambda x} \gamma_a(x) d\mu_t(x). \end{aligned}$$

Since γ_a satisfies the assumptions of Proposition 4.1,

$$\int_{-\infty}^{\infty} e^{-i\lambda x} \gamma_a(x) d\mu_t(x) = 2m(t) \int_{-\infty}^{\infty} e^{-itb} g_a(b + \lambda) \operatorname{Re} \int_B \hat{T}(ib) 1_A d\mu db.$$

By Proposition 4.3 together with the Fourier inversion formula $\int_{-\infty}^{\infty} e^{-i\lambda x} \gamma_a(x) dx = 2\pi g_a(\lambda)$,

$$\lim_{t \rightarrow \infty} \int_{-\infty}^{\infty} e^{-i\lambda x} \gamma_a(x) d\mu_t(x) = 2\pi d_\beta g_a(\lambda) \mu(A) \mu(B) = d_\beta \int_{-\infty}^{\infty} e^{-i\lambda x} \gamma_a(x) dx \mu(A) \mu(B).$$

Hence, we have shown that the hypothesis of Proposition 4.2 holds with $C = d_\beta \mu(A) \mu(B)$. It now follows from Proposition 4.2 with $I = [0, h]$ that

$$U_{A,B}(t+h) - U_{A,B}(t) = \mu_t([0, h]) \rightarrow d_\beta \mu(A) \mu(B) h,$$

as $t \rightarrow \infty$. ■

The proof of Propositions 4.1 and 4.3 are given in Section 5. They rely on standard arguments combined with the following estimates for \hat{T} . Let $\tilde{\ell}(t) = \int_1^t \ell(s) s^{-1} ds$. For $\beta \in (0, 1)$ define $c_\beta = i \int_0^\infty e^{-i\sigma} \sigma^{-\beta} d\sigma$.

Proposition 4.4 (a) When $\beta < 1$, there exists $\delta \in (0, 1)$ such that

$$\operatorname{Re} \hat{T}(s) = (\operatorname{Re} c_\beta^{-1}) \ell(1/|s|)^{-1} |s|^{-\beta} (P(0) + E(s)), \quad \text{for all } s \in \overline{\mathbb{H}} \cap B_\delta(0), s \neq 0,$$

where $P(0)v = \int_Y v d\mu$ for $v \in L^1$, and $E(s)$ is a family of operators satisfying $\lim_{s \rightarrow 0} \|E(s)\|_{\mathcal{B} \rightarrow L^1} = 0$.

When $\beta = 1$, there is a corresponding statement of the form

$$\operatorname{Re} \hat{T}(ib) = \frac{\pi}{2} \ell(1/|b|) \tilde{\ell}(1/|b|)^{-2} |b|^{-1} (P(0) + E(b)), \quad \text{for all } b \in \mathbb{R}, 0 < |b| < \delta.$$

(b) For any $L > 0$, there exists $C > 0$ such that for $0 < |b| \leq L$,

$$\|\operatorname{Re} \hat{T}(ib)\|_{\mathcal{B} \rightarrow L^1} \leq \begin{cases} C \ell(1/|b|)^{-1} |b|^{-\beta} & \beta < 1 \\ C \ell(1/|b|) \tilde{\ell}(1/|b|)^{-2} |b|^{-1} & \beta = 1 \end{cases}.$$

(c) For any $L > 0$, there exists $\gamma > 1 - \beta$, $C > 0$, such that for all $0 < h < b \leq L$,

$$\|\hat{T}(i(b+h)) - \hat{T}(ib)\|_{\mathcal{B} \rightarrow L^1} \leq C (\tilde{\ell}(1/b)^{-2} b^{-2\beta} \tilde{\ell}(1/h) h^\beta + b^{-\beta} h^\gamma), \quad \beta \leq 1.$$

Proof The arguments for $s \in B_\delta(0)$ are written out in [13, Section 4]. In particular this covers part (a). The details for the remainder of parts (b) and (c) are given in Appendix A. ■

5 Completion of the proof of Theorem 2.3

In this section, we give the proof of Propositions 4.1 and 4.3, thereby completing the proof of Theorem 2.3.

5.1 Proof of Proposition 4.1

For $n \geq 0$, the Fourier transform of the distribution $G_n(x) = \mu(\tau_n(y) \leq x, y \in A \cap F^{-n}B)$ is given by $\int_Y 1_A 1_B \circ F^n e^{ib\tau_n} d\mu = \int_B \hat{R}(-ib)^n 1_A d\mu$. Hence

$$\begin{aligned} \int_{-\infty}^{\infty} e^{ibx} dV_{A,B}(x) &= \operatorname{Re} \int_0^{\infty} e^{ibx} dU_{A,B}(x) \\ &= \sum_{n=0}^{\infty} \operatorname{Re} \int_B \hat{R}(ib)^n 1_A d\mu = \operatorname{Re} \int_B \hat{T}(ib) 1_A d\mu. \end{aligned}$$

Let γ and g be as in the statement of Proposition 4.1. Choose $L > 0$ such that $\operatorname{supp} h \in [-L, L]$. Let $\psi(b) = \begin{cases} \ell(1/|b|)^{-1}|b|^{-\beta} & \beta < 1 \\ \ell(1/|b|)\tilde{\ell}(1/|b|)^{-2}|b|^{-1} & \beta = 1 \end{cases}$. By Proposition 4.4(b), $|\hat{T}(ib)1_A|_1 \ll \psi(b)\|1_A\|_{\mathcal{B}}$ for $|b| \leq L$. Since ψ is integrable, it follows from Fubini's theorem that

$$\int_{-\infty}^{\infty} \gamma(x) dV_{A,B}(x) = \int_{-\infty}^{\infty} g(b) \operatorname{Re} \int_B \hat{T}(ib) 1_A d\mu db.$$

Let $g_1(b) = e^{-ibt}g(b+\lambda)$ and set $\gamma_1(x) = \int_{-\infty}^{\infty} e^{ibx} g_1(b) db = e^{-i\lambda(x-t)}\gamma(x-t)$. Replacing γ, g with γ_1, g_1 , we obtain the conclusion of Proposition 4.1.

5.2 Proof of Proposition 4.3

We follow the proof of [19, Theorem 1] (an adaptation of the argument in [21]).

Let $W(b) = \operatorname{Re} \int_B \hat{T}(ib) 1_A d\mu$. Fix $\omega > 1$ and write $\int_{-\infty}^{\infty} e^{-itb} g_a(b + \lambda) \operatorname{Re} \int_B \hat{T}(ib) 1_A d\mu db = I_1(t, \omega) + I_2(t, \omega)$ where

$$I_1(t, \omega) = \int_{-\omega/t}^{\omega/t} e^{-itb} g_a(b + \lambda) W(b) db, \quad I_2(t, \omega) = \int_{|b| > \omega/t} e^{-itb} g_a(b + \lambda) W(b) db.$$

Proposition 4.3 follows immediately from the estimates for $I_1(t, \omega)$ and $I_2(t, \omega)$ below. Set $c_{\beta}^{-1} = \pi/2$ when $\beta = 1$.

Lemma 5.1 $\lim_{t \rightarrow \infty} m(t)I_1(t, \omega) = 2g_a(\lambda) \operatorname{Re} c_{\beta}^{-1} \int_0^{\omega} b^{-\beta} \cos b db \mu(A)\mu(B)$, for all $\omega > 1$.

Proof It follows from the definition of g_a that $|g_a(b_1) - g_a(b_2)| \leq a^{-2}|b_1 - b_2|$. Hence

$$\begin{aligned} \left| I_1(t, \omega) - g_a(\lambda) \int_{-\omega/t}^{\omega/t} e^{-itb} W(b) db \right| &\leq \int_{-\omega/t}^{\omega/t} |g_a(b + \lambda) - g_a(\lambda)| |W(b)| db \\ &\leq 2a^{-2}\omega t^{-1} \int_0^{\omega/t} |W(b)| db. \end{aligned}$$

By Proposition 4.4(b), $\int_0^{\omega/t} |W(b)| db \ll \tilde{\ell}(t)^{-1} t^{-(1-\beta)} \|1_A\| = m(t)^{-1} \|1_A\|$ for $t > \omega/\delta$. Hence

$$\lim_{t \rightarrow \infty} m(t) I_1(t, \omega) = g_a(\lambda) \lim_{t \rightarrow \infty} m(t) \int_{-\omega/t}^{\omega/t} e^{-itb} W(b) db.$$

Next, by Proposition 4.4(a), there is a function $h : \mathbb{R} \rightarrow \mathbb{R}$ with $|h(b)| \leq C \|1_A\|$ and $\lim_{b \rightarrow 0} h(b) = \mu(A)\mu(B)$ such that for $\beta < 1$,

$$\begin{aligned} \int_{-\omega/t}^{\omega/t} e^{-itb} W(b) db &= 2 \int_0^{\omega/t} W(b) \cos tb db = 2 \operatorname{Re} c_\beta^{-1} \int_0^{\omega/t} \ell(1/b)^{-1} b^{-\beta} h(b) \cos tb db \\ &= 2 \operatorname{Re} c_\beta^{-1} t^{-(1-\beta)} \int_0^\omega \ell(t/b)^{-1} b^{-\beta} h(b/t) \cos b db. \end{aligned}$$

Hence

$$m(t) \int_{-\omega/t}^{\omega/t} e^{-itb} W(b) db = 2 \operatorname{Re} c_\beta^{-1} \int_0^\omega [\ell(t)/\ell(t/b)] b^{-\beta} h(b/t) \cos b db.$$

By the dominated convergence theorem,

$$\lim_{t \rightarrow \infty} m(t) \int_{-\omega/t}^{\omega/t} e^{-itb} W(b) db = 2 \operatorname{Re} c_\beta^{-1} \int_0^\omega b^{-\beta} \cos b db \mu(A)\mu(B),$$

and the result for $\beta < 1$ follows. The case $\beta = 1$ follows from a more complicated dominated convergence argument as detailed in the proof of [13, Lemma 3.2(a)]. ■

Lemma 5.2 *Let $\beta' \in (\frac{1}{2}, \beta)$. Then $\limsup_{t \rightarrow \infty} m(t) I_2(t, \omega) = O(\omega^{-(2\beta'-1)})$.*

Proof It follows from evenness of g_a and $W(b)$, together with the fact that $\operatorname{supp} g_a \in [-a, a]$, that

$$I_2(t, \omega) = \int_{b > \omega/t} [e^{-itb} g_a(b + \lambda) + e^{itb} g_a(b - \lambda)] W(b) db = \int_{\omega/t}^{a+|\lambda|} h(b) W(b) db,$$

where $h(b) = e^{-itb} g_a(b + \lambda) + e^{itb} g_a(b - \lambda)$. Continuing as on [19, p. 278] down as far as [19, Equation (5.14)], we obtain

$$\begin{aligned} |I_2(t, \omega)| &\leq a^{-1} \int_{(\omega-\pi)/t}^{\omega/t} |W(b + \pi/t)| db + \pi a^{-2} t^{-1} \int_{\omega/t}^{a+|\lambda|} |W(b)| db \\ &\quad + a^{-1} \int_{\omega/t}^{a+|\lambda|} |W(b + \pi/t) - W(b)| db. \end{aligned}$$

These are almost identical to the expressions in the proof of [13, Lemma 3.5] and are estimated in the same way using Proposition 4.4. ■

6 Proof of Theorems 2.4 and 2.6

In this section, we prove Theorem 2.4 by establishing separately an upper bound (Corollary 6.3) and a lower bound (Corollary 6.5). In the process of obtaining the upper bound, we prove Theorem 2.6.

6.1 Upper bound for \liminf

In this subsection, the only parts of (H) that are required are (i)–(iii) with $s \in \mathbb{R}^+$ in part (i). A simplified version of the argument used in the proof of Proposition 4.4(a) can be used to obtain

Proposition 6.1 *Assume the setting of Theorem 2.6. For $\sigma > 0$,*

$$\hat{T}(\sigma) = D_\beta' \tilde{\ell}(1/\sigma)^{-1} \sigma^{-\beta} (P(0) + E(\sigma)),$$

where $D_\beta' = \Gamma(1 - \beta)^{-1}$ for $\beta \in (0, 1)$ and $D_0' = D_1' = 1$, and $E(\sigma)$ is a family of operators satisfying $\lim_{\sigma \rightarrow 0} \|E(\sigma)\|_{\mathcal{B} \rightarrow L^1} = 0$. ■

We can now complete

Proof of Theorem 2.6 For $n \geq 0$, the real Laplace transform of the distribution $G_n(x) = \mu(\tau_n(y) \leq x, y \in A \cap F^{-n}B)$ is given by $\int_Y 1_A 1_B \circ F^n e^{-\sigma \tau_n} d\mu = \int_B \hat{R}(e^{-\sigma})^n 1_A d\mu$. Hence,

$$\int_{-\infty}^{\infty} e^{-\sigma t} dU_{A,B}(t) = \sum_{n=0}^{\infty} \int_B \hat{R}(e^{-\sigma})^n 1_A d\mu = \int_B \hat{T}(e^{-\sigma}) 1_A d\mu.$$

The conclusion follows from Proposition 6.1 by the continuous time version of Karata's Tauberian Theorem [11, Theorem 1.7.1]. ■

Lemma 6.2 *Assume the setting of Theorem 2.6 with $\beta \in (0, 1]$. Let $z : [0, \infty) \rightarrow [0, \infty)$ be integrable. Then*

$$\liminf_{t \rightarrow \infty} m(t) \int_0^t z(t-y) dU_{A,B}(y) \leq d_\beta \mu(A) \mu(B) \int_0^\infty z dx.$$

Proof This is proved in the same way as [19, Lemma 9] using Theorem 2.6. ■

Corollary 6.3 *Assume the setting of Theorem 2.6 with $\beta \in (0, 1]$. Then for any $h > 0$,*

$$\liminf_{t \rightarrow \infty} m(t) (U_{A,B}(t+h) - U_{A,B}(t)) \leq d_\beta \mu(A) \mu(B) h.$$

Proof Let $z = 1_{[0,h]}$. By Lemma 6.2,

$$\begin{aligned} \liminf_{t \rightarrow \infty} m(t)(U_{A,B}(t+h) - U_{A,B}(t)) &= \liminf_{t \rightarrow \infty} m(t+h) \int_0^{t+h} z(t+h-y) dU_{A,B}(y) \\ &\leq d_\beta \mu(A) \mu(B) \int_0^\infty z dx = d_\beta \mu(A) \mu(B) h, \end{aligned}$$

as required. \blacksquare

6.2 Lower bound for \liminf

We require the following local limit theorem. Recall that $c_\beta = i \int_0^\infty e^{-i\sigma} \sigma^{-\beta} d\sigma$. Let $q_\beta(t) = \frac{1}{2\pi} \int_{-\infty}^\infty e^{ibt} e^{-c_\beta |b|^\beta} db$.

Lemma 6.4 *Assume the setting of Theorem 2.4. Let $d_n > 0$ be an increasing sequence with $d_n \rightarrow \infty$ such that $n\mu(\tau > d_n) = n\ell(d_n)d_n^{-\beta} \rightarrow 1$, as $n \rightarrow \infty$. Then for any $h > 0$ there exists $e_n > 0$ with $\lim_{n \rightarrow \infty} e_n = 0$ such that for all $t > 0$, $n \geq 1$,*

$$\left| \mu(y \in A \cap F^{-n}B : \tau_n(y) \in [t, t+h]) - \frac{h}{d_n} q_\beta(t/d_n) \mu(A) \mu(B) \right| \leq \frac{e_n}{d_n}.$$

The proof of Lemma 6.4 combines arguments from [13] and [34] and is given for completeness in Appendix B. (A related argument [3, Theorem 6.3] based on [12] gives a similar conclusion but without the error term. As pointed out in [19], the full result requires proceeding as in [34].)

Corollary 6.5 *Assume the setting of Theorem 2.4. Then for any $h > 0$,*

$$\liminf_{t \rightarrow \infty} m(t)(U_{A,B}(t+h) - U_{A,B}(t)) \geq d_\beta \mu(A) \mu(B) h.$$

Proof Let $m \geq k \geq 0$. By (2.1) and Lemma 6.4,

$$\begin{aligned} U_{A,B}(t+h) - U_{A,B}(t) &\geq \sum_{n=k}^m \mu(y \in A \cap F^{-n}B : \tau_n(y) \in [t, t+h]) \\ &= \sum_{n=k}^m \frac{h}{d_n} q_\beta(t/d_n) \mu(A) \mu(B) + E_{k,m}, \end{aligned}$$

where $E_{k,m} = \sum_{n=k}^m e_n/d_n$.

Let $\kappa \in (1, 1/\beta)$. Then $d_n^{-1} = O(n^{-\kappa})$ and $E_{k,m} = O(\sup_{n \geq k} |e_n|) \rightarrow 0$ as $k \rightarrow \infty$.

Choosing $k = \lceil C_1 t^\beta / \ell(t) \rceil$ and $m = \lceil C_2 t^\beta / \ell(t) \rceil$, for fixed $C_2 > C_1 > 0$ and arguing word for word as in [19, Proof of eq. (7.2)], we obtain

$$\liminf_{t \rightarrow \infty} m(t)(U_{A,B}(t+h) - U_{A,B}(t)) \geq \mu(A) \mu(B) \int_{C_1}^{C_2} x^{-1/\beta} q_\beta(x^{-1/\beta}) dx.$$

Now let $C_1 \rightarrow 0$ and $C_2 \rightarrow \infty$ and use that $\int_0^\infty x^{-1/\beta} q_\beta(x^{-1/\beta}) dx = d_\beta$. \blacksquare

7 General class of observables

In this section, we extend mixing for semiflows, Corollary 3.1, to cover more general classes of observables. As well as being of interest in its own right, this is useful for the extension to flows in Section 8. Throughout, we suppose that we are in the setting of Corollary 3.1; in particular $\beta \in (\frac{1}{2}, 1]$ and (H) holds.

From now on we suppose that Y is a metric space with Borel probability measure μ and that F and τ are almost everywhere continuous. Let \mathcal{C} be a collection of measurable subsets $A \subset Y$ with $1_A \in \mathcal{B}$ such that

- (i) $\mu(\partial A) = 0$ for all $A \in \mathcal{C}$,
- (ii) $A_1 \cap A_2 \in \mathcal{C}$ for all $A_1, A_2 \in \mathcal{C}$,
- (iii) \mathcal{C} is a basis for the topology on Y .

In practice, we can often take \mathcal{C} to consist of all measurable sets $A \subset Y$ with $1_A \in \mathcal{B}$ and $\mu(\partial A) = 0$. This is the case for the examples in Section 9.

Proposition 7.1 *Let $\mathcal{C}' = \{A \times [a_1, a_2] \subset Y^\tau : A \in \mathcal{C}\}$. Let \mathcal{D} be the ring generated by \mathcal{C}' . Choose a sequence $H_n \in \mathcal{D}$, $n \geq 1$, such that $H_n \subset H_{n+1}$ and $\mu(Y^\tau \setminus \bigcup_n H_n) = 0$. Let $v : Y^\tau \rightarrow \mathbb{R}$ be bounded, almost every continuous, and supported in H_n for some n , and let $w \in L^1(Y^\tau)$. Then*

$$\lim_{t \rightarrow \infty} m(t) \int_{Y^\tau} v w \circ F_t d\mu^\tau = d_\beta \int_{Y^\tau} v d\mu^\tau \int_{Y^\tau} w d\mu^\tau. \quad (7.1)$$

Proof It is immediate that conditions (i)–(iii) for \mathcal{C} are inherited by the collection \mathcal{C}' of subsets of Y^τ . In addition

- (iv) $\mu(A) < \infty$ for $A \in \mathcal{C}'$, and there exist $A_1, A_2, \dots \in \mathcal{C}'$ such that $\mu(Y \setminus \bigcup A_n) = 0$.

These are the conditions listed in [26, pages 434–435].

By Corollary 3.1,

$$\lim_{t \rightarrow \infty} m(t) \mu^\tau(A \cap F_t^{-1} B) = d_\beta \mu^\tau(A) \mu^\tau(B), \quad (7.2)$$

for all $A \in \mathcal{C}'$ and all measurable rectangles $B \subset Y^\tau$. The argument now proceeds as in [26] with obvious modifications since only A is restricted to lie in \mathcal{C}' . (In [26], A and B both lie in \mathcal{C}' leading to additional restrictions on w .)

By [26, page 435], property (7.2) extends first to all $A \in \mathcal{D}$, and second to all measurable subsets $A \subset Y^\tau$ measurable such that $\mu(\partial A) = 0$ and $A \subset H_n$ for some n . The result now follows from [26, lower half of page 435] (approximating v by step functions involving admissible sets A , and approximating w by simple functions). \blacksquare

One possible choice for the sequence H_n is the following:

Corollary 7.2 Define $H_n = \{(y, u) \in Y \times [0, \infty) : \tau(y) - n \leq u \leq \tau(y)\}$, $n \geq 1$. Then (7.1) holds for all bounded and almost every continuous functions $v : Y^\tau \rightarrow \mathbb{R}$ supported in H_n for some n , and all $w \in L^1(Y^\tau)$.

Proof Let $\mathcal{C}'' = \mathcal{C}' \cup \{E_n, n \geq 1\}$ where \mathcal{C}' is the collection of rectangles in Proposition 7.1 and $E_n = \bigcup_{j=1}^n F_j^{-1}(Y \times [0, 1])$. Let $\mathcal{I} = \{C \cap E_n : C \in \mathcal{C}', n \geq 1\}$ and define $\mathcal{C}''' = \mathcal{C}'' \cup \mathcal{I}$. Then \mathcal{C}''' is closed under finite intersections, and hence conditions (i)–(iv) are satisfied by the collection \mathcal{C}''' . We claim that property (7.2) holds for all $A \in \mathcal{C}'''$. Certainly, the sets E_n lie in the ring generated by \mathcal{C}''' , and $H_n \subset E_n$, so the conclusion follows from [26] with E_n playing the role of H_n .

It remains to verify the claim. By Corollary 3.1, property (7.2) holds for all $A \in \mathcal{C}'$. By Remark 3.2, this holds also for the sets E_n . Finally, if $I \in \mathcal{I}$, then I is contained in one of the rectangles in \mathcal{C}' and $\mu^\tau(\partial I) = 0$. Hence 1_I is a bounded and almost everywhere continuous function supported in a rectangle in \mathcal{C}' . The claim follows from Proposition 7.1. \blacksquare

8 Mixing for infinite measure flows

In this section, we show how mixing for semiflows extends to mixing for flows.

8.1 Assumptions and disintegration

We suppose throughout that $F_t : Y^\tau \rightarrow Y^\tau$ is a suspension semiflow over a map $F : Y \rightarrow Y$ with nonintegrable almost every continuous roof function $\tau : Y \rightarrow \mathbb{R}^+$ satisfying $\text{ess inf } \tau > 1$ and $\mu(\tau > t) = \ell(t)t^{-\beta}$, $\beta \in (\frac{1}{2}, 1]$, and we assume that (H) holds. We also assume that there exists a collection \mathcal{C} of subsets of Y as in Section 7.

Let $X = Y \times N$ where Y and N are bounded metric space. Let $f(y, z) = (Fy, G(y, z))$ where $F : Y \rightarrow Y$ and $G : Y \times N \rightarrow N$ are continuous almost everywhere.

Define $\tau : X \rightarrow \mathbb{R}^+$ by setting $\tau(y, z) = \tau(y)$ and define the suspension $X^\tau = \{(x, u) \in X \times \mathbb{R} : 0 \leq u \leq \tau(x)\} / \sim$ where $(x, \tau(x)) \sim (fx, 0)$. The suspension flow $f_t : X^\tau \rightarrow X^\tau$ is given by $f_t(x, u) = (x, u + t)$ computed modulo identifications, with ergodic invariant measure $\mu_X^\tau = \mu_X \times \text{Lebesgue}$.

Under two additional assumptions (A1) and (A2) below, we show in Theorem 8.5 that Corollary 3.1 for the semiflow F_t applies equally to the flow f_t .

First, we assume contractivity along N :

(A1) $\lim_{n \rightarrow \infty} d(f^n(y, z), f^n(y, z')) = 0$ for all $z, z' \in N$ uniformly in $y \in Y$.

Let $\pi : X \rightarrow Y$ be the projection $\pi(y, z) = y$. This defines a semiconjugacy between f and F . There exists a unique f -invariant ergodic probability measure μ_X on X such that $\pi_*\mu_X = \mu$, see for instance [9, Section 6].

Recall that R denotes the transfer operator for $F : Y \rightarrow Y$.

Proposition 8.1 Fix $z_0 \in N$. Suppose $v \in C^0(X)$. Then the limit

$$\eta_y(v) = \lim_{n \rightarrow \infty} (R^n v_n)(y), \quad v_n(y) = v \circ f^n(y, z_0),$$

exists for almost every $y \in Y$ and defines a probability measure supported on $\pi^{-1}(y)$. Moreover $y \mapsto \eta_y(v) = \int_{\pi^{-1}(y)} v d\eta_y$ is integrable and $\int_X v d\mu_X = \int_Y \int_{\pi^{-1}(y)} v d\eta_y d\mu(y)$.

Proof See for instance [14, Proposition 3]. ■

Remark 8.2 The proof of [14, Proposition 3] shows that the sequence $R^n v_n$ is Cauchy in $L^\infty(Y)$. If the metric on Y can be chosen so that $R^n v_n$ is continuous for each n , then $\bar{v} \in C^0(Y)$. (In fact, it can often be shown that \bar{v} is Hölder when v is Hölder [14].)

Note that $X^\tau = Y^\tau \times N$. Given $v \in C^0(X^\tau)$, define

$$\bar{v} : Y^\tau \rightarrow \mathbb{R}, \quad \bar{v}(y, u) = \int_{x \in \pi^{-1}(y)} v(x, u) d\eta_y(x).$$

Then

$$\int_{X^\tau} v d\mu_X^\tau = \int_{Y^\tau} \bar{v}(y, u) d\mu^\tau(y, u).$$

We require the additional assumption:

(A2) The function $\bar{v} : Y^\tau \rightarrow \mathbb{R}$ is almost everywhere continuous.

Remark 8.3 If v is uniformly continuous, then for any $\epsilon > 0$ there exists $\delta < 0$ such that $|\bar{v}(y, u) - \bar{v}(y, u')| < \epsilon$ for all $(y, u), (y, u') \in Y^\tau$ with $|u - u'| < \delta$. This combined with Remark 8.2 shows that condition (A2) is easily satisfied in practice for a large class of observables $v \in C^0(X^\tau)$.

Remark 8.4 The set up in this section (skew product $X = Y \times N$, roof function τ constant in the N direction) is not very restrictive. Suppose that $T_t : M \rightarrow M$ is a smooth flow defined on a Riemannian manifold M and that Λ is a partially hyperbolic attractor, so there exists a continuous DT_t -invariant splitting $T_\Lambda M = E^s \oplus E^{cu}$ where E^s is uniformly contracting and dominates E^{cu} . By [7, Proposition 3.2, Theorem 4.2], the stable bundle E^s extends to a neighbourhood U of Λ and integrates to a T_t -invariant collection \mathcal{W}^s of stable leaves that topologically foliate U .

This means that we can choose a topological submanifold $X \subset M$ that is a cross-section to the flow T_t formed as a union of stable leaves, and automatically the roof function τ is constant along stable leaves. (This construction has been widely used recently [5, 6, 8, 10].) Assuming for convenience the existence of a global chart for \mathcal{W}^s , we obtain a Poincaré map $f : X \rightarrow X$ where $X = Y \times N$ with N playing

the role of the stable direction. Moreover, f has the desired skew product form $f(y, z) = (Fy, G(y, z))$, where $F : Y \rightarrow Y$ is defined by quotienting along the stable leaves, and condition (A1) is automatically satisfied. Also (A2) holds by Remark 8.2. Hence our set up holds in its entirety provided $F : Y \rightarrow Y$ and $\tau : Y \rightarrow \mathbb{Z}^+$ satisfy the required properties.

8.2 The mixing result

Choose a sequence H_n , $n \geq 1$, of subsets of Y^τ as in Proposition 7.1.

Theorem 8.5 *Suppose that $\mu(\tau > n) = \ell(n)n^{-\beta}$ where $\beta \in (\frac{1}{2}, 1]$. Let $v \in C^0(X^\tau)$ be supported in $C \times N$ where C is a closed subset of $\text{Int } H_n$ for some $n \geq 1$. Let $w \in C^0(X^\tau)$ be uniformly continuous and supported on a set of finite measure. Assume that (H), (A1) and (A2) hold. Then*

$$\lim_{t \rightarrow \infty} m(t) \int_{X^\tau} v w \circ f_t d\mu_X^\tau = d_\beta \int_{X^\tau} v d\mu_X^\tau \int_{X^\tau} w d\mu_X^\tau.$$

Proof Following [10], we define $w_s : Y^\tau \rightarrow \mathbb{R}$, $s > 0$, by setting

$$w_s(y, u) = \overline{w \circ f_s} = \int_{x \in \pi^{-1}(y)} w \circ f_s(x, u) d\eta_y(x).$$

Note that $\int_{Y^\tau} |w_s| d\mu^\tau \leq \int_{X^\tau} |w| \circ f_s d\mu_X^\tau = \int_{X^\tau} |w| d\mu_X^\tau$ so $w_s \in L^1(Y^\tau)$ for all s .

The semiconjugacy $\pi : X \rightarrow Y$ extends to a measure-preserving semiconjugacy $\pi^\tau : X^\tau \rightarrow Y^\tau$, $\pi^\tau(x, u) = (\pi x, u)$. Write $m(t) \int_{X^\tau} v w \circ f_t d\mu_X^\tau = I_1(s, t) + I_2(s, t)$ where

$$\begin{aligned} I_1(s, t) &= m(t) \int_{X^\tau} v w_s \circ \pi^\tau \circ f_{t-s} d\mu_X^\tau, \\ I_2(s, t) &= m(t) \int_{X^\tau} v (w \circ f_s - w_s \circ \pi^\tau) \circ f_{t-s} d\mu_X^\tau. \end{aligned}$$

For $t > s$,

$$I_1(s, t) = m(t) \int_{X^\tau} v w_s \circ F_{t-s} \circ \pi^\tau d\mu_X^\tau = m(t) \int_{Y^\tau} \bar{v} w_s \circ F_{t-s} d\mu^\tau.$$

Since \bar{v} is bounded and almost every continuous, supported in H_n , and $w_s \in L^1(Y^\tau)$, it follows from Proposition 7.1 that for all $s > 0$,

$$\lim_{t \rightarrow \infty} I_1(s, t) = d_\beta \int_{Y^\tau} \bar{v} d\mu^\tau \int_{Y^\tau} w_s d\mu^\tau = d_\beta \int_{X^\tau} v d\mu_X^\tau \int_{X^\tau} w d\mu_X^\tau.$$

Choose $\psi : Y^\tau \rightarrow [0, 1]$ continuous such that $\text{supp } v \subset \text{supp } \psi \times N \subset H_n \times N$. Define

$$D_s : Y^\tau \rightarrow \mathbb{R}, \quad D_s(y, u) = \text{diam } w \circ f_s((\pi^\tau)^{-1}(y, u)).$$

Note that $|D_s| \leq 2|w|_\infty$ and $\mu^\tau(\text{supp } D_s) \leq \mu_X^\tau(f_s^{-1} \text{supp } w) = \mu_X^\tau(\text{supp } w) < \infty$, so $D_s \in L^1(Y^\tau)$. Also, $|w \circ f_s(x, u) - w_s \circ \pi^\tau(x, u)| \leq D_s \circ \pi^\tau(x, u)$. Hence for $t > s$,

$$|I_2(s, t)| \leq |v|_\infty m(t) \int_{X^\tau} \psi \circ \pi^\tau D_s \circ \pi^\tau \circ f_{t-s} d\mu_X^\tau = |v|_\infty m(t) \int_{Y^\tau} \psi D_s \circ F_{t-s} d\mu_Y^\tau.$$

Since $\psi \in C^0(Y^\tau)$ is supported in H_n and $D_s \in L^1(Y^\tau)$, it again follows from Proposition 7.1 that for all $s > 0$,

$$\limsup_{t \rightarrow \infty} I_2(s, t) \leq |v|_\infty d_\beta \int_{Y^\tau} \psi d\mu^\tau \int_{Y^\tau} D_s d\mu^\tau.$$

By uniform continuity of w and (A1), $\lim_{s \rightarrow \infty} |D_s|_\infty = 0$. Hence $|D_s|_1 \leq |D_s|_\infty \mu^\tau(\text{supp } D_s) \leq |D_s|_\infty \mu_X^\tau(\text{supp } w) \rightarrow 0$ as $s \rightarrow \infty$. This combined with the estimates for I_1 and I_2 yields the desired result. \blacksquare

9 Examples

Consider the map $g : [0, 1] \rightarrow [0, 1]$ given by

$$g(x) = x(1 + c_1 x^{\gamma_1}) \bmod 1 \quad \text{where } \gamma_1 \in [1, 2), c_1 \in (0, 1].$$

This is an example of an AFN map [39], namely a nonuniformly expanding one-dimensional map with at most countably (in this case finitely) many branches with finite images and satisfying Adler's distortion condition $\sup |g''|/|g'|^2 < \infty$. Up to scaling, there is a unique absolutely continuous invariant measure μ_0 . The measure μ_0 is infinite and the density has a singularity at the neutral fixed point 0.

Let $\tau_0 : [0, 1] \rightarrow [1, \infty)$ be a roof function of bounded variation and Hölder continuous, and let g_t denote the suspension semiflow on $[0, 1]^{\tau_0}$ with invariant measure $\mu_0^{\tau_0} = \mu_0 \times \text{Lebesgue}$. Note that there is now a neutral periodic orbit of period $\tau_0(0)$.

In [13], under a Dolgopyat-type condition on τ_0 and for sufficiently regular observables v and w supported away from the neutral periodic orbit, we proved a mixing result with rates and higher order asymptotics. Here we obtain the mixing result without requiring the Dolgopyat-type condition or high regularity for the observables. It suffices that g_t has two periodic orbits (other than the neutral periodic orbit) whose periods have irrational ratio. Let $\beta = 1/\gamma_1 \in (\frac{1}{2}, 1]$ and define

$$m(t) = \begin{cases} \log t & \beta = 1 \\ t^{1-\beta} & \beta \in (\frac{1}{2}, 1) \end{cases}. \quad \text{We show that}$$

$$\lim_{t \rightarrow \infty} m(t) \int_{[0,1]} v w \circ g_t d\mu_0 = \text{const} \int_{[0,1]} v d\mu_0 \int_{[0,1]} w d\mu_0, \quad (9.1)$$

where the constant depends only on g . Here v is any continuous functions supported away from the neutral periodic point and w is any integrable function.

Remark 9.1 If $c_1 = 1$, then f is Markov and is a special case of the class of maps considered by [37]. In this case, it suffices that τ_0 is Hölder continuous.

We have restricted to the case $\gamma_1 < 2$. If $\gamma_1 \geq 2$, then Corollary 3.5 applies. For all $\gamma \geq 1$, Corollary 3.3 applies.

Proof of the mixing property (9.1)

The first step is to pass from the original suspension semiflow on $[0, 1]^{\tau_0}$ to a suspension of the form Y^τ where (Y, μ) is a probability space and τ is a nonintegrable roof function.

We take Y to be the interval of domain of the rightmost branch of g . Define the first return map $F = g^\sigma : Y \rightarrow Y$ where $\sigma = \min\{n \geq 1 : g^n y \in Y\}$. Then $\mu = (\mu_0|Y)/\mu_0(Y)$ is an absolutely continuous invariant probability measure for F . Define the induced roof function $\tau : Y \rightarrow \mathbb{R}^+$ given by $\tau(y) = \sum_{\ell=0}^{\sigma(y)-1} \tau_0(g^\ell y)$. Let $F_t : Y^\tau \rightarrow Y^\tau$ be the corresponding suspension semiflow with infinite invariant measure μ^τ .

Since τ_0 is Hölder, it is standard that $\mu(\tau > t) \sim c_0 t^{-\beta}$ where c_0 is a positive constant (see for example [13, Proposition 7.1]).

We take \mathcal{B} to be the space of bounded variation functions on Y , with norm $\|v\|_{BV} = |v|_1 + \text{Var } v$. Then \mathcal{B} is compactly embedded in L^1 and embedded in L^∞ , so (H)(iii) is satisfied with $p = \infty$. Also, (H)(ii) is standard for mixing AFN maps.

In the next subsection, we show how to verify (H)(i) and (H)(iv). It then follows from Corollary 7.2 that mixing for F_t holds for all continuous v supported in $H_n = \{(y, u) \in Y \times [0, \infty) : \tau(y) - n \leq u \leq \tau(y)\}$ for some $n \geq 1$, and all $w \in L^1(Y^\tau)$.

The projection $\pi : Y^\tau \rightarrow [0, 1]^{\tau_0}$, $\pi(y, u) = g_u(y, 0)$, defines a measure-preserving semiconjugacy from $F_t : Y^\tau \rightarrow Y^\tau$ to $g_t : [0, 1]^{\tau_0} \rightarrow [0, 1]^{\tau_0}$. Let $v, w : [0, 1]^{\tau_0} \rightarrow \mathbb{R}$ be observables where v is bounded and almost everywhere continuous supported away from the neutral periodic orbit, and w is integrable. Define the lifted observables $v = v \circ \pi$, $w = w \circ \pi : Y^\tau \rightarrow \mathbb{R}$. Then $v \circ \pi$ is a bounded and almost everywhere continuous function supported in H_n for some n , and $w \circ \pi$ is integrable. This completes the proof of (9.1).

Conditions (H)(i) and (H)(iv)

As promised, we verify the remaining parts of (H).

The Lasota-Yorke condition (H)(i) We use Remark 2.2. Parts (a) and (b) are done in [13, Proposition 7.2]. Here we verify Remark 2.2(c).

Since the density $d\mu/d\text{Leb}$ lies in BV and is bounded above and below, it suffices to work with the non-normalised transfer operator $\hat{P}(ib)v = P(e^{ib\tau}v)$ where $\int_Y Pvw d\text{Leb} = \int_Y vw \circ F d\text{Leb}$.

Let $\lambda = \inf g|_Y > 1$. Fix $L > 0$. We claim that there exists a constant C' such that

$$\|\hat{P}(ib)^n v\|_{BV} \leq C' n |v|_1 + C' n \lambda^{-n} \text{Var } v,$$

for all $|b| \leq L$, $n \geq 1$, $v \in BV$. Part (c) follows.

It remains to prove the claim. Let $n \geq 1$ and let $\{I\}$ be the partition of domains of branches for P^n . There is a constant C_0 independent of n such that $\sup_I 1/(F^n)' \leq C_0 \text{diam } I$ for all I . Also $F' \geq \lambda$, so $|1/(F^n)'| \leq 1/\lambda^n$ for all n .

Write

$$\hat{P}(ib)^n v = \sum_I \{\zeta_n e^{ib\tau_n} v\} \circ \psi_I 1_{F^n I},$$

where $\zeta_n = 1/(F^n)'$, ψ_I is the inverse branch $(F^n|_I)^{-1}$, and $\tau_n = \sum_{j=0}^{n-1} \tau \circ f^n$ (not to be confused with τ_0). We have the standard estimate

$$\begin{aligned} |\hat{P}(ib)^n v|_1 &\leq |\hat{P}(ib)^n v|_\infty \leq \sum_I \sup_I (\zeta_n |v|) \leq \sum_I \sup_I \zeta_n (\inf_I |v| + \text{Var}_I v) \\ &\leq \sum_I \sup_I \zeta_n (\text{diam } I)^{-1} \int_I |v| + \sum_I \lambda^{-n} \text{Var}_I v \leq C_0 |v|_1 + \lambda^{-n} \text{Var } v. \end{aligned}$$

Next,

$$\begin{aligned} \text{Var}(\hat{P}(ib)^n v) &\leq \sum_I \text{Var}_I(\zeta_n e^{ib\tau_n} v) + 2 \sum_I \sup_I (\zeta_n |v|) \\ &\leq \sum_I \text{Var}_I(\zeta_n v) + \sum_I \sup_I (\zeta_n |v|) \text{Var}_I e^{ib\tau_n} + 2C_0 |v|_1 + 2\lambda^{-n} \text{Var } v. \end{aligned}$$

A standard argument shows that

$$\sum_I \text{Var}_I(\zeta_n v) \leq C_1 |v|_1 + \lambda^{-n} \text{Var } v,$$

where $C_1 = \sup_n |(F^n)''|/[(F^n)']^2$. Also,

$$\text{Var}_I e^{ib\tau_n} \leq |b| \text{Var}_I \tau_n \leq L \sum_{j=0}^{n-1} \text{Var}_I(\tau \circ F^j) = L \sum_{j=0}^{n-1} \text{Var}_{F^j I} \tau.$$

Let a be the domain of a branch for F . Then $\tau|_a = \sum_{\ell=0}^{\sigma(a)-1} \tau_0 \circ g^\ell$. Since the images $g^\ell a$ are disjoint for $\ell < \sigma(a)$, it follows that $\text{Var}_a \tau \leq \text{Var } \tau_0$. But $F^j I$ lies in such a domain a , so $\text{Var}_{F^j I} \tau \leq \text{Var } \tau_0$ and it follows that $\text{Var}_I e^{ib\tau_n} \leq Ln \text{Var } \tau_0$. Hence

$$\sum_I \sup_I (\zeta_n |v|) \text{Var}_I e^{ib\tau_n} \leq Ln \text{Var } \tau_0 \sum_I \sup_I (\zeta_n |v|) \leq Ln \text{Var } \tau_0 (C_0 |v|_1 + \lambda^{-n} \text{Var } v).$$

Combining these estimates we have shown that $\|\hat{P}(ib)^n v\|_{BV} \leq (3C_0 + C_1 + C_0 L \text{Var } \tau_0) n |v|_1 + (4 + L \text{Var } \tau_0) n \lambda^{-n} \text{Var } v$ proving the claim.

The aperiodicity condition (H)(iv) Passing to the L^2 adjoint of $\hat{R}(ib)$, it is equivalent to rule out the possibility that there exists $b \neq 0$ and a BV eigenfunction $v : Y \rightarrow S^1$ such that $e^{ib\tau} v \circ F = v$. Suppose that $y \in Y$ is a periodic point of period k for F . Now, BV functions have one-sided limits, and F is orientation preserving, so $v(y+) = v(F^k(y+))$. Substituting into the equation $e^{ib\tau_k} v \circ F^k = v$ we obtain $e^{ibq} = 1$ where $q = \tau_k(y+)$ is the period of the corresponding periodic orbit for g_t .

Hence if g_t has two periodic orbits (other than the neutral periodic orbit) whose periods have irrational ratio, then (H)(iv) holds.

Remark 9.2 Combining this example with Remark 8.4 leads to examples of partially hyperbolic intermittent flows preserving an infinite measure. See [29, 30] for similar examples in the discrete time invertible setting. In addition to extending to continuous time, our examples are an improvement over those in [29, 30] as far as mixing is concerned, since we require no assumptions on smoothness of foliations (in contrast to [29]) or Markov structure (in contrast to [30]).

A Proof of Proposition 4.4

In this appendix, we complete the proof of Proposition 4.4. As already indicated, by [13, Lemma 4.9] it suffices to consider parts (b) and (c) with $b \in [\delta, L]$.

Proposition A.1 *Let $L > 0$. There exists $\gamma > 1 - \beta$, $C > 0$, such that for all $0 < b < b + h \leq L$,*

$$\|\hat{R}(i(b+h)) - \hat{R}(ib)\|_{\mathcal{B} \rightarrow L^1} \leq C h^\gamma.$$

Proof Fix $a > 0$ such that $e^{iba} \neq 1$ for $b \in (0, L]$. Slightly modifying the arguments in [13, Section 4.2] (where $a = 1$), we define

$$M(t)v = R(1_{\{t < \tau < t+a\}}v), \quad \hat{M}(s) = \int_0^\infty M(t)e^{-st} dt, \quad g(s) = \frac{s}{e^{sa} - 1}.$$

As in [13, Lemma 4.3], we obtain that $\hat{R}(ib) = g(ib)\hat{M}(ib)$ for $b \in (0, L]$. Following [13, Proposition 4.4, Lemma 4.5], there exists $\gamma > 1 - \beta$ such that

$$|\hat{M}(ib)v|_1 \ll \|v\|_{\mathcal{B}}, \quad |(\hat{M}(i(b+h)) - \hat{M}(ib))v|_1 \ll h^\gamma \|v\|_{\mathcal{B}},$$

for all $b, h > 0$. Finally, $|g(ib)| \ll 1$ and $|g(i(b+h)) - g(ib)| \ll h$ for $0 < b < b+h \leq L$, and the result follows. \blacksquare

By hypothesis (H) and Proposition A.1, it follows from [25, Theorem 1] that for each $b > 0$ and all $\gamma' \in (1 - \beta, \gamma)$ there exists $h_0 > 0$ and $C > 0$ such that $\|\hat{T}(i(b+h)) - \hat{T}(ib)\|_{\mathcal{B} \rightarrow L^1} \leq Ch^{\gamma'}$ for all $|h| < h_0$. By compactness of $[\delta, L]$ there exists $\gamma' > 1 - \beta$, $C > 0$, such that $\|\hat{T}(ib)\|_{\mathcal{B} \rightarrow L^1} \leq C$ and $\|\hat{T}(i(b+h)) - \hat{T}(ib)\|_{\mathcal{B} \rightarrow L^1} \leq Ch^{\gamma'}$ for all $\delta \leq b < b+h < L$. This completes the proof of Proposition 4.4.

B Proof of the local limit theorem

In this section, we give the proof of Lemma 6.4. Let

$$\mu_n(I) = \mu(y \in A \cap F^{-n}B : \tau_n(y) \in I).$$

Define

$$V_n(t, h, a) = \int_{-\infty}^{\infty} K_a(t - t') \mu_n([d_n t', d_n(t' + h)]) dt',$$

where

$$K_a(x) = a^{-1} K(a^{-1}x), \quad K(x) = \frac{1}{2\pi} \sin^2 \frac{1}{2}x.$$

Lemma B.1 *Let $L > 0$. Then*

$$V_n(t, h, a) = h \{ q_\beta(t) \mu(A) \mu(B) + e(n, h, a, t) \} \quad \text{for } a \geq (Ld_n)^{-1},$$

where $e(n, h, a, t) \rightarrow 0$ as $n \rightarrow \infty$, $h \rightarrow 0$ and $a \rightarrow 0$, uniformly in $t \in \mathbb{R}$.

Proof In fact, we show that

$$|V_n(t, h, a) - h q_\beta(t) \mu(A) \mu(B)| \leq \text{const. } h \{ e_1(n) + e_2(h) + e_3(a) \}$$

where $\lim_{n \rightarrow \infty} e_1(n) = \lim_{h \rightarrow 0} e_2(h) = \lim_{a \rightarrow 0} e_3(a) = 0$.

By (H)(ii) and (H)(iv), we can write $\hat{R}(ib) = \lambda(b)P(b) + Q(b)$ for $|b| \leq \delta$, where $\lambda(b)$ is a continuous family of simple eigenvalues for $\hat{R}(ib) : \mathcal{B} \rightarrow \mathcal{B}$ with $\lambda(0) = 1$, and $P(b)$ is the associated spectral projection with complementary projection $Q = I - P$. Hence

$$\hat{R}(ib)^n = \lambda(b)^n P(0) + \lambda(b)^n (P(b) - P(0)) + Q(b)^n. \quad (\text{B.1})$$

Moreover, there exist constants $C > 0$, $\gamma > 1 - \beta$, $\alpha_1 \in (0, 1)$, where

$$\|P(b) - P(0)\|_{\mathcal{B} \rightarrow L^1} \leq C|b|^\gamma, \quad \|Q(b)^n\|_{\mathcal{B}} \leq C\alpha_1^n, \quad \text{for all } |b| \leq \delta, n \geq 1. \quad (\text{B.2})$$

(The first of these estimates follows from [13, Lemma 4.6] using (H).)

Also, by (H), we can choose $C > 0$, $\alpha_1 \in (0, 1)$ so that

$$\|\hat{R}(ib)^n\|_{\mathcal{B}} \leq C\alpha_1^n \quad \text{for all } b \in [\delta, L], n \geq 1, \quad (\text{B.3})$$

(see, for instance, [25, Corollary 2, part 2]).

Using that $\mu(\tau > t) = \ell(t)t^{-\beta}$, we obtain from the proof of [13, Lemma 4.7] that $1 - \lambda(b) \sim c_\beta \ell(1/|b|)b^\beta$. Hence

$$\lambda(b) \sim e^{-c_\beta \ell(1/|b|)|b|^\beta} \text{ as } b \rightarrow 0, \quad \lim_{n \rightarrow \infty} \lambda(d_n^{-1}b)^n = e^{-c_\beta |b|^\beta}. \quad (\text{B.4})$$

By (B.4) and Potter's bounds, there exists $C_1, C_2 > 0$ and $\beta' \in (0, \beta)$ such that

$$|\lambda(d_n^{-1}b)|^n \leq C_1 e^{-C_2|b|^{\beta'}} \quad \text{for all } |b| \leq \delta d_n, n \geq 1. \quad (\text{B.5})$$

Let $k(b) = \begin{cases} (1 - |b|), & |b| < 1 \\ 0, & |b| \geq 1 \end{cases}$ and define $k_a(b) = k(ab)$. Then $k_a(b) = \int_{\mathbb{R}} e^{ixb} K_a(x) dx$. We compute that

$$\begin{aligned} V_n(t, h, a) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-ib(t-t')} k_a(b) db \int_{A \cap F^{-n}B} 1_{\{\tau_n \in [d_n t', d_n(t'+h)]\}} d\mu dt' \\ &= \frac{1}{2\pi} \int_{|b| \leq a^{-1}} e^{-ibt} k_a(b) \int_{A \cap F^{-n}B} \int_{d_n^{-1}\tau_n - h}^{d_n^{-1}\tau_n} e^{ibt'} dt' d\mu db \\ &= \frac{1}{2\pi} \int_{|b| \leq a^{-1}} e^{-itb} k_a(b) (1 - e^{-ihb}) (ib)^{-1} \int_{A \cap F^{-n}B} e^{id_n^{-1}b\tau_n} d\mu db \\ &= \frac{h}{2\pi} \int_{|b| \leq a^{-1}} e^{-itb} g(b, h, a) \int_B \hat{R}(id_n^{-1}b)^n 1_A d\mu db, \end{aligned}$$

where $g(b, h, a) = k_a(b) (1 - e^{-ihb}) (ihb)^{-1}$.

Note that $|g(b, h, a)| \leq 1$. Using (B.3) and that $a \geq (Ld_n)^{-1}$,

$$\begin{aligned} \left| \int_{\delta d_n \leq |b| \leq a^{-1}} e^{-itb} g(b, h, a) \int_B \hat{R}(id_n^{-1}b)^n 1_A d\mu db \right| &\leq \|1_A\|_{\mathcal{B}} \int_{\delta d_n \leq |b| \leq Ld_n} \|\hat{R}(id_n^{-1}b)^n\|_{\mathcal{B}} db \\ &= \|1_A\|_{\mathcal{B}} d_n \int_{\delta \leq |b| \leq L} \|\hat{R}(ib)^n\|_{\mathcal{B}} db \leq C \|1_A\|_{\mathcal{B}} d_n \alpha_1^n. \end{aligned}$$

Hence this term can be incorporated into $e_1(n)$.

It remains to analyse

$$\frac{h}{2\pi} \int_{|b| \leq \delta d_n} e^{-itb} g(b, h, a) \int_B \hat{R}(id_n^{-1}b)^n 1_A d\mu db = \frac{h}{2\pi} (I_1 + I_2 + I_3),$$

where by (B.1),

$$\begin{aligned} I_1 &= \int_{|b| \leq \delta d_n} e^{-itb} g(b, h, a) \int_B \lambda(d_n^{-1}b)^n P(0) 1_A d\mu db, \\ I_2 &= \int_{|b| \leq \delta d_n} e^{-itb} g(b, h, a) \int_B \lambda(d_n^{-1}b)^n (P(d_n^{-1}b) - P(0)) 1_A d\mu db, \\ I_3 &= \int_{|b| \leq \delta d_n} e^{-itb} g(b, h, a) \int_B Q(d_n^{-1}b)^n 1_A d\mu db. \end{aligned}$$

By (B.2) and (B.5),

$$|I_2| \leq \int_{|b| \leq \delta d_n} C_1 e^{-C_2|b|^{\beta'}} C |d_n^{-1}b|^{\gamma} \|1_A\|_{\mathcal{B}} db \leq CC_1 \|1_A\|_{\mathcal{B}} d_n^{-\gamma} \int_{-\infty}^{\infty} |b|^{\gamma} e^{-C_2|b|^{\beta'}} db \ll d_n^{-\gamma},$$

and

$$|I_3| \leq d_n \int_{|b| \leq \delta} C \alpha_1^n \|1_A\|_{\mathcal{B}} db \ll d_n \alpha_1^n.$$

Again, these terms can be incorporated into $e_1(n)$.

This leaves the term $I_1 = I'_1 \mu(A) \mu(B)$ where $I'_1 = \int_{|b| \leq \delta d_n} e^{-itb} g(b, h, a) \lambda(d_n^{-1}b)^n db$. Write $I'_1 = J_1 + J_2 + J_3$ where

$$\begin{aligned} J_1 &= \int_{|b| \leq \delta d_n} e^{-itb} k_a(b) \{(1 - e^{-ihb})(ihb)^{-1} - 1\} \lambda(d_n^{-1}b)^n db, \\ J_2 &= \int_{|b| \leq \delta d_n} e^{-itb} (k_a(b) - 1) \lambda(d_n^{-1}b)^n db, \\ J_3 &= \int_{|b| \leq \delta d_n} e^{-itb} \lambda(d_n^{-1}b)^n db. \end{aligned}$$

Since $|(1 - e^{-ihb})(ihb)^{-1} - 1| \leq \frac{1}{2}h|b|$ it follows from (B.5) that

$$|J_1| \leq h \int_{-\infty}^{\infty} C_1 e^{-C_2|b|^{\beta'}} |b| db \ll h.$$

Also,

$$|J_2| \leq \int_{-\infty}^{\infty} |k_a(b) - 1| C_1 e^{-C_2|b|^{\beta'}} db,$$

which converges to zero by the dominated convergence theorem as $a \rightarrow 0$. These are the sole contributions to e_2 and e_3 respectively.

Finally,

$$|J_3 - 2\pi q_{\beta}(t)| \leq \int_{|b| \leq \delta d_n} |\lambda(d_n^{-1}b)^n - e^{-c_{\beta}|b|^{\beta}}| db + \int_{|b| \geq \delta d_n} e^{-c_{\beta}|b|^{\beta}} db,$$

which converges to zero by (B.4), (B.5) and the dominated convergence theorem as $n \rightarrow \infty$. ■

Lemma B.2 *Let $\epsilon > 0$ and $L > 0$. There exists $n_0 \geq 1$ and $h_0 > 0$ such that*

$$h(q_{\beta}(t)\mu(A)\mu(B) - \epsilon) \leq \mu_n([d_n t, d_n(t+h)]) \leq h(q_{\beta}(t)\mu(A)\mu(B) + \epsilon),$$

for all $n \geq n_0$, $h \in [(Ld_n)^{-1}, h_0]$, $t \in \mathbb{R}$.

Proof Let $\tilde{q}_{\beta} = q_{\beta}\mu(A)\mu(B)$. Since q_{β} is the Fourier transform of an L^1 function, \tilde{q}_{β} is uniformly continuous and bounded. Let $q_{\infty} = |\tilde{q}_{\beta}|_{\infty}$ and choose $h_1 \in (0, 1)$ such that $|\tilde{q}_{\beta}(t) - \tilde{q}_{\beta}(t')| \leq \frac{1}{4}\epsilon$ whenever $|t - t'| \leq h_1$.

For $\epsilon_1 > 0$, set $\epsilon_2 = \int_{|x| > 1/\epsilon_1} K(x) dx$. We choose $\epsilon_1 \in (0, \frac{1}{6})$ sufficiently small that

$$(q_{\infty} + 2\epsilon_1 q_{\infty} + \frac{1}{2}\epsilon)(1 - \epsilon_2)^{-1} - q_{\infty} \leq \epsilon, \quad 2\epsilon_1 q_{\infty} + \epsilon_2(q_{\infty} + \epsilon) \leq \frac{1}{2}\epsilon. \quad (\text{B.6})$$

By Lemma B.1, there exists $n_0 \geq 1$ and $h_0 \in (0, h_1)$ such that for all $n \geq n_0$, $h \in [(Ld_n)^{-1}, h_0]$, $t \in \mathbb{R}$,

$$\begin{aligned} V_n(t - \epsilon_1 h, h(1 + 2\epsilon_1), \epsilon_1^2 h) &\leq h(1 + 2\epsilon_1) \tilde{q}_\beta(t - \epsilon_1 h) + \frac{1}{6} \epsilon h \\ &\leq h(1 + 2\epsilon_1) (\tilde{q}_\beta(t) + \frac{1}{4} \epsilon) + \frac{1}{6} \epsilon h \leq h(\tilde{q}_\beta(t) + 2\epsilon_1 q_\infty + \frac{1}{2} \epsilon), \end{aligned} \quad (\text{B.7})$$

where we used the constraint $\epsilon_1 \leq \frac{1}{6}$. Also, we can ensure that

$$\begin{aligned} V_n(t + \epsilon_1 h, h(1 - 2\epsilon_1), \epsilon_1^2 h) &\geq h(1 - 2\epsilon_1) \tilde{q}_\beta(t + \epsilon_1 h) - \frac{1}{4} \epsilon h \\ &\geq h(1 - 2\epsilon_1) (\tilde{q}_\beta(t) - \frac{1}{4} \epsilon) - \frac{1}{4} \epsilon h \geq h(\tilde{q}_\beta(t) - 2\epsilon_1 q_\infty - \frac{1}{2} \epsilon). \end{aligned} \quad (\text{B.8})$$

Now, for $|t'| \leq \epsilon_1 h$,

$$\begin{aligned} \mu_n([d_n(t + \epsilon_1 h - t'), d_n(t - \epsilon_1 h - t' + h)]) &\leq \mu_n([d_n t, d_n(t + h)]) \\ &\leq \mu_n([d_n(t - \epsilon_1 h - t'), d_n(t + \epsilon_1 h - t' + h)]). \end{aligned}$$

Also $\int_{-\infty}^{\infty} K dx = 1$, so

$$1 - \epsilon_2 = \int_{|x| \leq 1/\epsilon_1} K(x) dx = \epsilon_1^2 h \int_{|x| \leq 1/\epsilon_1} K_{\epsilon_1^2 h}(\epsilon_1^2 h x) dx = \int_{|x| \leq \epsilon_1 h} K_{\epsilon_1^2 h}(x) dx.$$

Hence

$$\begin{aligned} &V_n(t - \epsilon_1 h, h(1 + 2\epsilon_1), \epsilon_1^2 h) \\ &= \int_{-\infty}^{\infty} K_{\epsilon_1^2 h}(t') \mu_n([d_n(t - \epsilon_1 h - t'), d_n(t + \epsilon_1 h - t' + h)]) dt' \\ &\geq \int_{|t'| \leq \epsilon_1 h} K_{\epsilon_1^2 h}(t') \mu_n([d_n(t - \epsilon_1 h - t'), d_n(t + \epsilon_1 h - t' + h)]) dt' \\ &\geq \int_{|t'| \leq \epsilon_1 h} K_{\epsilon_1^2 h}(t') \mu_n([d_n t, d_n(t + h)]) dt' = (1 - \epsilon_2) \mu_n([d_n t, d_n(t + h)]). \end{aligned}$$

By (B.6) and (B.7),

$$\begin{aligned} \mu_n([d_n t, d_n(t + h)]) &\leq (1 - \epsilon_2)^{-1} V_n(t - \epsilon_1 h, h(1 + 2\epsilon_1), \epsilon_1^2 h) \\ &\leq h(\tilde{q}_\beta(t) + 2\epsilon_1 q_\infty + \frac{1}{2} \epsilon) (1 - \epsilon_2)^{-1} \leq h(\tilde{q}_\beta(t) + \epsilon). \end{aligned}$$

Arguing similarly, and exploiting the last estimate for $\mu_n([d_n t, d_n(t + h)])$,

$$\begin{aligned} &V_n(t + \epsilon_1 h, h(1 - 2\epsilon_1), \epsilon_1^2 h) \\ &\leq \int_{|t'| \leq \epsilon_1 h} K_{\epsilon_1^2 h}(t') \mu_n([d_n(t + \epsilon_1 h - t'), d_n(t - \epsilon_1 h - t' + h)]) dt' \\ &\quad + \int_{|t'| \geq \epsilon_1 h} K_{\epsilon_1^2 h}(t') h(q_\infty + \epsilon) dt' \\ &\leq \mu_n([d_n t, d_n(t + h)]) + \epsilon_2 h(q_\infty + \epsilon). \end{aligned}$$

By (B.6) and (B.8),

$$\begin{aligned} \mu_n([d_n t, d_n(t + h)]) &\geq V_n(t + \epsilon_1 h, h(1 - 2\epsilon_1), \epsilon_1^2 h) - \epsilon_2 h(q_\infty + \epsilon) \\ &\geq h((\tilde{q}_\beta(t) - 2\epsilon_1 q_\infty - \frac{1}{2} \epsilon - \epsilon_2(q_\infty + \epsilon)) \geq h(\tilde{q}_\beta(t) - \epsilon). \end{aligned}$$

This completes the proof. ■

Proof of Lemma 6.4 After a change of variables, Lemma B.2 reads as follows:

Let $\epsilon > 0$ and $L > 0$. There exists $n_0 \geq 1$ and $h_0 > 0$ such that

$$\sup_{t \in \mathbb{R}} d_n \left| \mu_n([t, t+h]) - \frac{h}{d_n} q_\beta(d_n^{-1}t) \mu(A) \mu(B) \right| \leq h\epsilon, \quad (\text{B.9})$$

for all $n \geq n_0$, $h \in [L^{-1}, d_n h_0]$.

Fix $h > 0$ and define $e_n = \sup_{t \in \mathbb{R}} d_n \left| \mu_n([t, t+h]) - \frac{h}{d_n} q_\beta(d_n^{-1}t) \mu(A) \mu(B) \right|$. We must show that $\lim_{n \rightarrow \infty} e_n = 0$.

Let $L = 1/h$. By (B.9), for any $\epsilon > 0$ there exists $n_0 \geq 1$, $h_0 > 0$, such that $e_n \leq h\epsilon$ for all $n \geq n_0$ subject to the constraint $d_n h_0 \geq h$. Since $d_n \rightarrow \infty$, there exists $n_1 \geq n_0$ such that $d_n h_0 \geq h$ for all $n \geq n_1$. Hence $e_n \leq h\epsilon$ for all $n \geq n_1$ as required. ■

Acknowledgements This research began as a result of discussions with Jon Aaronson, Henk Bruin, Dima Dolgopyat, Péter Nándori, Françoise Pène and Doma Szász at the thematic program *Mixing Flows and Averaging Methods* at the Erwin Schrödinger Institute (ESI), Vienna, April/May 2016. We are particularly grateful to Bruin, Dolgopyat, Nándori and Pène for continued discussions on this topic.

The research of DT and IM was supported in part by funding from ESI. The research of IM was supported in part by a European Advanced Grant *StochExtHomog* (ERC AdG 320977).

References

- [1] J. Aaronson. *An Introduction to Infinite Ergodic Theory*. Math. Surveys and Monographs **50**, Amer. Math. Soc., 1997.
- [2] J. Aaronson. Rational weak mixing in infinite measure spaces. *Ergodic Theory Dynam. Systems* **33** (2013) 1611–1643.
- [3] J. Aaronson and M. Denker. Local limit theorems for partial sums of stationary sequences generated by Gibbs-Markov maps. *Stoch. Dyn.* **1** (2001) 193–237.
- [4] T. M. Adams and C. E. Silva. Weak rational ergodicity does not imply rational ergodicity. *Israel J. Math.* **214** (2016) 491–506.
- [5] V. Araújo, O. Butterley and P. Varandas. Open sets of axiom A flows with exponentially mixing attractors. *Proc. Amer. Math. Soc.* **144** (2016) 2971–2984.
- [6] V. Araújo and I. Melbourne. Exponential decay of correlations for nonuniformly hyperbolic flows with a $C^{1+\alpha}$ stable foliation, including the classical Lorenz attractor. *Ann. Henri Poincaré* **17** (2016) 2975–3004.

- [7] V. Araújo and I. Melbourne. Existence and smoothness of the stable foliation for sectional hyperbolic attractors. *Bull. London Math. Soc.* (2017). To appear.
- [8] V. Araújo, I. Melbourne and P. Varandas. Rapid mixing for the Lorenz attractor and statistical limit laws for their time-1 maps. *Comm. Math. Phys.* **340** (2015) 901–938.
- [9] V. Araujo, M. J. Pacifico, E. R. Pujals and M. Viana. Singular-hyperbolic attractors are chaotic. *Trans. Amer. Math. Soc.* **361** (2009) 2431–2485.
- [10] A. Avila, S. Gouëzel and J. Yoccoz. Exponential mixing for the Teichmüller flow. *Publ. Math. Inst. Hautes Études Sci.* (2006) 143–211.
- [11] N. H. Bingham, C. M. Goldie and J. L. Teugels. *Regular variation*. Encyclopedia of Mathematics and its Applications **27**, Cambridge University Press, Cambridge, 1987.
- [12] L. Breiman. *Probability*. Addison-Wesley Publishing Company, Reading, Mass.-London-Don Mills, Ont., 1968.
- [13] H. Bruin, I. Melbourne and D. Terhesiu. Rates of mixing for nonMarkov infinite measure semiflows. Preprint, 2016.
- [14] O. Butterley and I. Melbourne. Disintegration of invariant measures for hyperbolic skew products. *Israel J. Math.* (2017). To appear.
- [15] F. Caravenna and R. A. Doney. Local large deviations and the strong renewal theorem. Preprint, 2016.
- [16] D. Dolgopyat. Prevalence of rapid mixing in hyperbolic flows. *Ergodic Theory Dynam. Systems* **18** (1998) 1097–1114.
- [17] D. Dolgopyat and P. Nándori. Private communication.
- [18] R. A. Doney. One-sided local large deviation and renewal theorems in the case of infinite mean. *Probab. Theory Related Fields* **107** (1997) 451–465.
- [19] K. B. Erickson. Strong renewal theorems with infinite mean. *Trans. Amer. Math. Soc.* **151** (1970) 263–291.
- [20] W. Feller and S. Orey. A renewal theorem. *J. Math. Mech.* **10** (1961) 619–624.
- [21] A. Garsia and J. Lamperti. A discrete renewal theorem with infinite mean. *Comment. Math. Helv.* **37** (1962/1963) 221–234.
- [22] S. Gouëzel. Correlation asymptotics from large deviations in dynamical systems with infinite measure. *Colloq. Math.* **125** (2011) 193–212.

- [23] J. Kautzsch, M. Kesseböhmer and T. Samuel. On the convergence to equilibrium of unbounded observables under a family of intermittent interval maps. *Ann. Henri Poincaré* **17** (2016) 2585–2621.
- [24] J. Kautzsch, M. Kesseböhmer, T. Samuel and B. O. Stratmann. On the asymptotics of the α -Farey transfer operator. *Nonlinearity* **28** (2015) 143–166.
- [25] G. Keller and C. Liverani. Stability of the spectrum for transfer operators. *Annali della Scuola Normale Superiore di Pisa, Classe di Scienze* **XXVIII** (1999) 141–152.
- [26] K. Krickeberg. Strong mixing properties of Markov chains with infinite invariant measure. *Proc. Fifth Berkeley Sympos. Math. Statist. and Probability (Berkeley, Calif., 1965/66), Vol. II: Contributions to Probability Theory, Part 2*, Univ. California Press, Berkeley, Calif., 1967, pp. 431–446.
- [27] M. Lenci. On infinite-volume mixing. *Comm. Math. Phys.* **298** (2010) 485–514.
- [28] M. Lenci. Uniformly expanding Markov maps of the real line: exactness and infinite mixing. Preprint, 2014.
- [29] C. Liverani and D. Terhesiu. Mixing for some non-uniformly hyperbolic systems. *Ann. Henri Poincaré* **17** (2016) 179–226.
- [30] I. Melbourne. Mixing for invertible dynamical systems with infinite measure. *Stoch. Dyn.* **15** (2015) 1550012, 25.
- [31] I. Melbourne and D. Terhesiu. Operator renewal theory and mixing rates for dynamical systems with infinite measure. *Invent. Math.* **189** (2012) 61–110.
- [32] I. Melbourne and D. Terhesiu. Operator renewal theory for continuous time dynamical systems with finite and infinite measure. *Monatsh. Math.* **182** (2017) 377–431.
- [33] K. Petersen. *Ergodic Theory*. Cambridge Studies in Adv. Math. **2**, Cambridge Univ. Press, 1983.
- [34] C. Stone. A local limit theorem for nonlattice multi-dimensional distribution functions. *Ann. Math. Statist.* **36** (1965) 546–551.
- [35] D. Terhesiu. Improved mixing rates for infinite measure preserving systems. *Ergodic Theory Dynam. Systems* (2015) 585–614.
- [36] D. Terhesiu. Mixing rates for intermittent maps of high exponent. *Probab. Theory Related Fields* **166** (2016) 1025–1060.

- [37] M. Thaler. Transformations on $[0, 1]$ with infinite invariant measures. *Israel J. Math.* **46** (1983) 67–96.
- [38] M. Thaler. The asymptotics of the Perron-Frobenius operator of a class of interval maps preserving infinite measures. *Studia Math.* **143** (2000) 103–119.
- [39] R. Zweimüller. Ergodic structure and invariant densities of non-Markovian interval maps with indifferent fixed points. *Nonlinearity* **11** (1998) 1263–1276.